The error of the Galerkin method for a nonhomogeneous Kirchhoff type wave equation

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Abstract—The paper deals with the boundary value problem for a nonlinear integro-differential equation describing the dynamic state of a beam. To approximate the solution with respect to a spatial variable, the Galerkin method is used, the error of which is estimated. At the end of the paper a difference-iteration technique of solving the Galerkin system is presented.

Keywords—Nonlinear beam equation, approximate algorithm, Galerkin method, error estimate.

I. PROBLEM FORMULATION

Let us consider the nonlinear differential equation

\[
\frac{\partial^2 w}{\partial t^2}(x,t) + \frac{\partial^4 w}{\partial x^4}(x,t) \\
- \left( \alpha + \beta \int_0^t \left( \frac{\partial w}{\partial \xi}(\xi,t) \right)^2 d\xi \right) \frac{\partial^2 w}{\partial x^2}(x,t) \\
= f(x,t), \quad 0 < x < L, \quad 0 < t \leq T,
\]

with the initial boundary conditions

\[
w(x,0) = w^0(x), \quad \frac{\partial w}{\partial t}(x,0) = w'(x), \\
w(0,t) = w(L,t) = 0, \\
\frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(L,t) = 0, \\
0 \leq x \leq L, \quad 0 \leq t \leq T,
\]

where \(\alpha, \beta, L\) and \(T\) are some positive constants, \(f(x,t)\), \(w^0(x)\), \(w'(x)\) are the given functions and \(w(x,t)\) is the function we want to define.

II. BACKGROUND OF THE PROBLEM

Equation (1) describes the oscillation of a beam. The corresponding homogeneous equation was obtained by Woinowsky-Krieger [27] in 1950.

The nonlinear term in the brackets is the correction to the classical Euler-Bernoulli equation

\[
w_{tt} + c^2 w_{xxxx} = 0,
\]

where the tension changes induced by the vibration of the beam during deflection are not taken into account. This nonlinear term was for the first time proposed by Kirchhoff [13] who generalized d’Alembert’s classical model. Therefore equation (1) is often called a Kirchhoff type equation for a dynamic beam. Note that Arosio [1] calls the function of the integral \(\int_0^L w_\xi^2 d\xi\) the Kirchhoff correction (briefly, the K-correction) and makes a reasonable statement that the K-correction is inherent in a lot of physical phenomena.

The works dealing with the mathematical aspects of equation (1) when \(f(x,t) = 0\) and its generalization

\[
w_{tt} + w_{xxxx} - M \left( \int_0^L w_\xi^2 d\xi \right) w_{xx} = f(x,t,w),
\]

\(M(\lambda) \geq \text{const} > 0,\)

as well as some modifications of the above equations belong to Ball [2, 3], Biler [5], Brito [6], Dickey [10], Guo and Guo [12], Kouemou-Patcheu [14], Medeiros [17], Menezes et al. [18], Panizzi [20], Pereira [25] and to others. The subject of investigation concerned the questions of the existence and uniqueness of a solution [2, 3, 12, 14, 17, 18, 20, 25], its asymptotic behaviour [5, 6, 10, 14], stabilization and control problems [12] and so on.

The topic of an approximate solution of Kirchhoff equations, which the present paper is concerned with, was treated by Choo and Chung [7], Choo et al. [8], Clark et al. [9], Geveci and Christie [11]. Speaking more exactly, the finite difference and finite element approximate solutions are investigated and the corresponding error estimates are derived in [7, 8]. Numerical analysis of solutions for a beam with moving boundary is carried out in [9]. The question of the...
stability and convergence of a semi-discrete and fully discrete approximation is dealt with in [11]. The problem of an approximate solution of a static Kirchhoff equation was studied by Ma [16] and Tsai [26].

Approximate methods for other equations containing the \( K \)-correction or being reduced to equations with it are investigated in [22, 23, 24].

III. ASSUMPTIONS

Suppose that the initial functions are represented in the form

\[
w'(x) = \sum_{i=1}^{\infty} a_i^{(l)} \sin \frac{i \pi}{L} x, \quad l = 0,1, \quad 0 \leq x \leq L,\]

and

\[
a_i^{(0)2} \leq \frac{\omega_0}{i^{p+4}}, \quad a_i^{(l)2} \leq \frac{\omega_i}{i^p}, \quad i = 1,2,\ldots, \quad (4)
\]

where \( p, \omega_0, \omega_i \) are some positive numbers and also \( p > 1 \).

Assume that

\[
f(x,t) \in C(0,T;L_2(0,L)). \quad (5)
\]

Suppose that there exists a solution of problem (1), (2) which is represented in the form

\[
w(x,t) = \sum_{i=1}^{\infty} w_i(t) \sin \frac{i \pi}{L} x, \quad (6)
\]

where the coefficients \( w_i(t) \) satisfy the following infinite system of differential equations

\[
w_i''(t) + \left( \frac{\pi a_i}{L} \right)^4 w_i(t) = f_i(t),
\]

\[
f_i(t) = \frac{2}{L} \left[ \int_0^t f(x,t) \sin \frac{i \pi}{L} x dx, \quad i = 1,2,\ldots, \quad 0 < t \leq T,\right.
\]

with the initial conditions

\[
w_i(0) = a_i^{(0)}, \quad w'_i(0) = a_i^{(1)}, \quad i = 1,2,\ldots. \quad (8)
\]

Assume also that

\[
\sum_{i=1}^{\infty} w_i'^{2}(t) \quad \text{and} \quad \sum_{i=1}^{\infty} j^4 w_i^{2}(t) \quad (9)
\]

converge.

IV. THE GALERKIN APPROXIMATION

Let us perform approximation of the solution with respect to the variable \( x \). For this we use the Galerkin method. A solution will be sought in the form of a finite series

\[
w_n(x,t) = \sum_{i=1}^{n} w_m(t) \sin \frac{i \pi}{L} x, \quad (10)
\]

where the coefficients \( w_m(t) \) are solutions of the system of differential equations

\[
w_m''(t) + \left( \frac{\pi a_i}{L} \right)^4 w_m(t) + \left( \frac{\pi a_i}{L} \right)^4 \left( \alpha + \beta \frac{\pi^2}{2L} \sum_{j=1}^{n} j^2 w_m^2(t) \right) w_m(t) = f_i(t), \quad i = 1,2,\ldots, \quad 0 < t \leq T, \quad (11)
\]

with the initial conditions

\[
w_m(0) = a_i^{(0)}, \quad w'_m(0) = a_i^{(1)}, \quad i = 1,2,\ldots, n. \quad (12)
\]

Now we are going to estimate the error of the Galerkin method. To achieve this aim it is necessary to introduce several notions and to prove some auxiliary statements. Let \( \lambda \) and \( \mu \) be \( n \)-dimensional vectors, \( \lambda = (\lambda_i)_{i=1}^{n}, \quad \mu = (\mu_i)_{i=1}^{n}. \)

In the first place, we define respectively the scalar product and the norm

\[
(\lambda,\mu)_n = \sum_{i=1}^{n} \lambda_i \mu_i, \quad \| \lambda \|_n = (\lambda,\lambda)_n. \quad (13.1)
\]

Next, using the functions \( w_m(t), f_i(t) \) and the coefficients \( a_i^{(l)}, i = 1,2,\ldots,n, \quad l = 0,1 \), from (10), (7) and (3) we form the vectors

\[
w_n(t) = (w_m(t))_{i=1}^{n}, \quad f_n(t) = (f_i(t))_{i=1}^{n}, \quad a_n^l = (a_i^{(l)})_{i=1}^{n}, \quad l = 0,1. \quad (13.2)
\]

We also define the matrix and the energetic norm
Using this notation, (11), (12) can be written in the vector form

\[
\mathbf{w}_n(t) + Q_n^4 \mathbf{w}_n(t) + \left( \alpha + \beta \frac{L}{2} \| \mathbf{w}_n(t) \|_{Q_n^2}^2 \right) Q_n^2 \mathbf{w}_n(t) = \mathbf{f}_n(t),
\]

with the boundary conditions

\[
\mathbf{w}_n(0) = a_n^0, \quad \mathbf{w}_n'(0) = a_n^1.
\]

V. THE ERROR OF THE GALERKIN METHOD

By the coefficients of decomposition (6) we form the vector

\[
p_n(t) = (w_i(t))_{i=1}^n.
\]

By the error of the Galerkin method we understand the difference between the vectors \( \mathbf{w}_n(t) \) and \( p_n(t) \)

\[
\Delta \mathbf{w}_n(t) = \mathbf{w}_n(t) - p_n(t).
\]

Let us derive an equation for the error.

Using (16) and (13), the first \( n \) equations of system (7) and the first \( n \) equalities from each of the initial conditions (8) are written in the form

\[
\left( \Delta \mathbf{w}_n(t) \right)'' + Q_n^4 \Delta \mathbf{w}_n(t)
\]

\[
+ \left( \alpha + \beta \frac{L}{2} \| \Delta \mathbf{w}_n(t) \|_{Q_n^2}^2 \right) Q_n^2 \Delta \mathbf{w}_n(t)
\]

\[
+ z_n(t) = \mathbf{f}_n(t),
\]

where \( z_n(t) \) is the vector defined by the formula

\[
z_n(t) = \beta \frac{\pi^2}{2L} \left( \sum_{i=n-1}^{\infty} \int^t_0 w_i^2(\tau) \right)^{1/2} \left( \sum_{i=1}^{\infty} \int^t_0 w_i^2(\tau) \right)^{1/2}.
\]

Subtracting (18) and (19) from (14) and (15), respectively, and taking into account (17), we write the equation for the error

\[
\mathbf{w}_n(t)'' + Q_n^4 \mathbf{w}_n(t)
\]

\[
+ \left( \alpha + \beta \frac{L}{2} \| \mathbf{w}_n(t) \|_{Q_n^2}^2 \right) Q_n^2 \mathbf{w}_n(t)
\]

\[
+ \mathbf{f}_n(t) = \mathbf{w}_n(t),
\]

and denote

\[
\Phi(t) = \sum_{i=1}^{\infty} w_i^2(t) + \left( \frac{\pi}{L} \right)^4 \sum_{i=1}^{\infty} i^4 w_i^2(t)
\]

\[
+ \frac{1}{\beta L} \left( \alpha + \beta \frac{\pi^2}{2L} \sum_{i=1}^{\infty} i^2 w_i^2(t) \right)^2,
\]

then the result is written as \( \Phi'(t) = 2 \sum_{i=1}^{\infty} f_i(t) w_i'(t) \), which means that for \( 0 < t \leq T \) we have

\[
\Phi(t) \leq \Phi(0) + 2 \sup_{0 \leq \tau \leq T} \left[ \int_{\tau}^{\infty} f_i(t) \right] \left[ \sum_{i=1}^{\infty} w_i^2(\tau) \right]^{1/2} d\tau.
\]

Taking (24) into account we infer that

\[
\Phi(t) \leq \Phi(0) + 2 \sup_{0 \leq \tau \leq T} \left[ \int_{\tau}^{\infty} f_i(t) \right] \left[ \sum_{i=1}^{\infty} w_i^2(\tau) \right]^{1/2} d\tau.
\]

We need to use in (25) the following Bellman and Bihari generalization of Gronwall’s inequality [4].

Let \( \gamma : [0, \infty) \to [0, \infty) \) be a continuous function and
$z : (0, \infty) \to (0, \infty)$ be a nondecreasing continuous function.

Then the inequality $y(t) \leq c + \int_0^t z(y(s)) ds$, $0 \leq t < \infty$,
where $c$ is a positive constant, implies $y(t) \leq Z^{-1}(Z_0) < \infty$,
$0 \leq t < Z_0$, for a positive number $Z_0$ smaller than $Z(\infty)$.

Here

$Z(t) = \int_c^t \frac{d \tau}{z(\tau)}$, $t \geq c$.

In the case under consideration

$y(t) = \Phi(t)$, $c = \Phi(0)$, $z(\tau) = m \tau^\frac{1}{2}$,

$m = 2 \left( \max_{0 \leq t \leq T} \sum_{i=1}^\infty f_i^2(t) \right)^{\frac{1}{2}}$, $Z_0 = T$.

Thus

$Z(t) = 2 \left( t^\frac{1}{2} - c^\frac{1}{2} \right)$, $Z^{-1}(t) = \left( c^\frac{1}{2} + \frac{m}{2} t \right)^2$.

As a result we obtain

$\Phi(t) \leq \left( \Phi(0) + T \left( \sup_{0 \leq t \leq T} \sum_{i=1}^\infty f_i^2(t) \right)^{\frac{1}{2}} \right)^2$. (26)

By (26), (24) and the relations

$\|p_n w(t)\|_{Q^2_t} \leq \sum_{i=1}^\infty w_i^2(t)$,

$\|p_n w(t)\|_{Q^l_t} \leq \sum_{i=1}^\infty \left( \frac{n_i}{L} \right)^2 w_i^2(t)$, $l = 1, 2$,

$L \int_0^t f_i^2(x, t) dx = \frac{L}{2} \sum_{i=1}^\infty f_i^2(t)$

which follow from (16), (13), (7) and (5), we see that

$\left( \|p_n w(t)\|_{Q^2_t} \right)^2 \leq \sum_{i=1}^\infty \left( \frac{n_i}{L} \right)^2 w_i^2(t)$,

and then verify the fulfillment of (23) for $l = 2$, where $c_0$ is defined by (28).

Lemma 2. The inequality

$\|w_n(t)\|_{Q^2_t} \leq c_2$, (31)

where the value $c_2$ does not depend on $t$, is valid.

Proof. Multiplying (14) scalarly by $2w_n'(t)$, we obtain

$\Phi_n'(t) = 2(f_n(t), w_n'(t))_n$, where

$\Phi_n'(t) = \left( \Phi_n(0) + T \left( \sup_{0 \leq t \leq T} \int_0^t f_n^2(x, t) dx \right)^{\frac{1}{2}} \right)^2$. (28)

Let us calculate $\Phi(0)$. Using (24), (8), (3) and (4) we get

$\Phi(0) = \sum_{i=1}^\infty a_i^{(i)2} + \left[ \frac{\pi}{L} \right]^4 \sum_{i=1}^\infty i^4 a_i^{(i)2}$

$+ \frac{1}{\beta L} \left[ \alpha + \beta \frac{\pi^2}{2L} \sum_{i=1}^\infty i^2 a_i^{(i)2} \right]^2$ (29)

From (27), first taking into account that by virtue of (13)

$\|p_t w(t)\|_{Q^2_t} \geq \frac{\pi}{L} \|p_t w(t)\|_{Q^2_t}$ we obtain (23) for $l = 1$, where

$c_1 = 2 \frac{1}{\beta L} \left[ \left( \frac{\pi}{L} \right)^4 + 2 \alpha \left( \frac{\pi}{L} \right)^2 + c_0 \beta L \right] \left( \frac{\pi}{L} \right)^2$ (30)

and then verify the fulfillment of (23) for $l = 2$, where $c_0$ is defined by (28). $\square$
\[ \Phi_n(t) = \sum_{i=1}^{n} a_i(t)^2 + \left( \frac{\pi}{L} \right)^4 \sum_{i=1}^{n} i \beta^2 a_i(t)^2 \]

Therefore we get the relation

\[ \Phi_n(t) \leq \Phi_n(0) + 2 \int_0^t \| f_n(t) \|_m \| w_n(t) \|_d \, \tau. \] (33)

Let us apply the Bellman-Bihari inequality and definition (32) to (23). We have

\[ y(t) = \Phi_n(0), \quad c = \Phi_n(0), \quad z(t) = m \tau^\frac{1}{2}, \]

\[ m = 2 \sup_{0 \leq \tau \leq T} \| f_n(t) \|_m, \quad Z_0 = T. \]

Therefore as above

\[ Z(t) = \frac{2}{m} \left( t^\frac{1}{2} - c^\frac{1}{2} \right)^2, \quad Z^{-1}(t) = \left( c^\frac{1}{2} + \frac{m}{2} t \right)^2. \]

Hence we conclude that

\[ \Phi_n(t) \leq \left( \Phi_n^2(0) + T \sup_{0 \leq \tau \leq T} \| f_n(t) \|_m \right)^\frac{1}{2}. \]

This relation which together with (32), (13) and (7) imply the fulfillment of (31) with

\[ c_2 = 2 \frac{1}{\beta L} \left[ \left( \frac{\pi}{L} \right)^4 + 2 \alpha \left( \frac{\pi}{L} \right)^2 + c_3 \beta L \right] \]

\[ - \left( \frac{\pi}{L} \right)^2 + \alpha \right], \]

where

\[ c_3 = \left( \Phi_n^2(0) + T \sup_{0 \leq \tau \leq T} \sum_{i=1}^{n} f_i^2(t) \right)^2, \] (35)

and give the inequality

\[ \Phi_n(t) \leq c_3 \] (36)

to be used below. □

If it is required to calculate or estimate \( c_2 \), we may use the following formulas for \( \Phi_n(0) \)

\[ \Phi_n(0) = \sum_{i=1}^{n} a_i(0)^2 + \left( \frac{\pi}{L} \right)^4 \sum_{i=1}^{n} i \beta^2 a_i(0)^2 \]

\[ + \frac{1}{\beta L} \left[ \alpha + \beta \frac{\pi^2}{2L} \sum_{i=1}^{n} i^2 a_i(0)^2 \right]^2 \leq \Phi(0), \] (37)

which are the result of the application of (32), (15), (13) together with (4), (3), (29). Besides the integral test for the convergence of series is used, by which

\[ \sum_{i=1}^{n} \frac{1}{i^{p+1}} \leq 1 + \int_1^n \frac{1}{x^{p+1}} \, dx, \quad l = 0, 2. \]

Applying (30), (34)-(36), (28) and (7), we observe that

\[ c_2 \leq c_1. \] (38)

**Lemma 3.** The inequality

\[ \| z_n(t) \|_m \leq \frac{c_4}{n^{p-1}}, \] (39)

where the value \( c_4 \) does not depend on \( t \), is valid.

**Proof.** From (20) and (13) it follows that

\[ \| z_n(t) \|_m = \beta \frac{\pi^2}{2L} \sum_{i=n+1}^{\infty} i^2 \| w_i(t) \|_{d^4}. \] (40)

Using (9), let us introduce into consideration the function

\[ \Psi_n(t) = \sum_{i=n+1}^{\infty} w_i^2(t) + \left( \frac{\pi}{L} \right)^4 \sum_{i=n+1}^{\infty} i^4 w_i^2(t) \]

\[ + \left( \frac{\pi}{L} \right)^2 \left( \alpha + \beta \frac{\pi^2}{2L} \sum_{i=n+1}^{\infty} i^2 w_i^2(t) \right) \sum_{i=n+1}^{\infty} i^2 w_i^2(t). \] (41)
After multiplying the equation in (7) by \(2w_j(t)\) and summing the resulting equality over \(i = n + 1, n + 2, \ldots\), we obtain

\[
\Psi_n'(t) = \beta\pi^3 \sum_{j=1}^{\infty} j^3 w_j(t)w_j'(t) \sum_{i=n+1}^{\infty} i^2 w_i^2(t). \tag{42}
\]

By (24), (26), (28) and (7) we have

\[
\sum_{j=1}^{\infty} j^2 w_j(t)w_j'(t) \leq \frac{1}{2} \left( \frac{L}{\pi} \right)^2 \left[ \sum_{j=1}^{\infty} w_j^2(t) + \left( \frac{\pi}{L} \right)^4 \sum_{j=1}^{\infty} j^4 w_j^2(t) \right] \tag{43}
\]

\[
\leq \frac{1}{2} \left( \frac{L}{\pi} \right)^2 \Phi(t) \leq \frac{1}{2} c_0 \left( \frac{L}{\pi} \right)^2.
\]

Further, comparing the sum \(\sum\limits_{i=n+1}^{\infty} i^2 w_i^2(t)\) from (40) with the function \(\Psi_n(t)\) from (41), we infer

\[
\sum_{i=n+1}^{\infty} i^2 w_i^2(t) \leq \left( \frac{L}{\pi} \right)^2 \left( \alpha + \left( \frac{\pi}{L} \right)^2 \right)^{-1} \Psi_n(t). \tag{44}
\]

By virtue of (42)-(44) and the Gronwall inequality

\[
\Psi_n(t) \leq \Psi_n(0) \exp \left[ \frac{1}{2} c_0 \beta L \left( \alpha + \left( \frac{\pi}{L} \right)^2 \right)^{-1} t \right]. \tag{45}
\]

We need to estimate \(\Psi_n(0)\). This estimate is obtained by using (41), (8), (4), (3) and the formula

\[
\sum_{i=n+1}^{\infty} \frac{1}{i^{p+1}} \leq \int_{n}^{\infty} \frac{1}{x^{p+1}} \, dx, \quad l = 0, 2,
\]

which follows from the integral test for the convergence of series. As a result we have

\[
\Psi_n(0) = \sum_{i=n+1}^{\infty} a_i^{(12)} + \left( \frac{\pi}{L} \right)^4 \sum_{j=n+1}^{\infty} j^4 a_j^{(02)}
+ \left( \frac{\pi}{L} \right)^2 \left( \alpha + \frac{\pi^2}{2L} \sum_{j=1}^{\infty} j^2 a_j^{(02)} \right) \sum_{i=n+1}^{\infty} i^2 a_i^{(02)}
\leq \left( \omega_1 + \omega_0 \left( \frac{\pi}{L} \right)^4 \right) \sum_{i=n+1}^{\infty} \frac{1}{i^{p+1}}
+ \omega_0 \left( \frac{\pi}{L} \right)^2 \left[ \alpha + \beta \int_0^t \left( \frac{\pi}{L} \right)^2 \left( w^{(0)}(x) \right)^2 \, dx \right] \sum_{i=n+1}^{\infty} \frac{1}{i^{p+1}} \tag{46}
\]

By (24), (26), (28) and (7) we have

\[
\sum_{j=1}^{\infty} j^2 w_j(t)w_j'(t) \leq \frac{1}{2} \left( \frac{L}{\pi} \right)^2 \Phi(t) \leq \frac{1}{2} c_0 \left( \frac{L}{\pi} \right)^2.
\]

Further, comparing the sum \(\sum\limits_{i=n+1}^{\infty} i^2 w_i^2(t)\) from (40) with the function \(\Psi_n(t)\) from (41), we infer

\[
\sum_{i=n+1}^{\infty} i^2 w_i^2(t) \leq \left( \frac{L}{\pi} \right)^2 \left( \alpha + \left( \frac{\pi}{L} \right)^2 \right)^{-1} \Psi_n(t). \tag{44}
\]

By virtue of (42)-(44) and the Gronwall inequality

\[
\Psi_n(t) \leq \Psi_n(0) \exp \left[ \frac{1}{2} c_0 \beta L \left( \alpha + \left( \frac{\pi}{L} \right)^2 \right)^{-1} t \right]. \tag{45}
\]

Let us formulate the main result.

**Theorem.** The inequality

\[
\left\| \left( \Delta w_n(t) \right)' \right\|_{Q^2}^2 + \left\| \Delta w_n(t) \right\|_{Q^2}^2 + c(t) \left\| \Delta w_n(t) \right\|_{Q^2}^2 \leq \frac{c(t)}{n^{p+1}},
\]

where \(c(t) \) is defined below, is fulfilled for the error of the Galerkin method.

**Proof.** After the scalar multiplication of (21) by \(2(\Delta w_n(t))'\) we obtain

\[
\left\|
\begin{array}{l}
\left( \Delta w_n(t) \right)' \\
\Delta w_n(t)
\end{array}
\right\|^2_{Q^2} + c(t) \left\|
\begin{array}{l}
\Delta w_n(t)
\end{array}
\right\|^2_{Q^2} \leq \frac{c(t)}{n^{p+1}},
\]

where \(c(t) \) is defined below, is fulfilled for the error of the Galerkin method.
\[ F_n'(t) = \frac{1}{2} BL \left( \| \Delta w_n(t) \|_{Q^t_n}^2 + \| w_n(t) \|_{Q^t_n}^2 \right) \]
\[ + 2 \left( \| p_n w(t) \|_{Q^t_n}^2 - \| w_n(t) \|_{Q^t_n}^2 \right) \]
\[ \times \left( Q_n^2 p_n w(t), (\Delta w_n(t)) \right)_{Q^t_n} \]
\[ + 2 \left( z_n(t), (\Delta w_n(t)) \right)_{Q^t_n}, \]
where

\[ F_n(t) = \left\| \Delta w_n(t) \right\|_{Q^t_n}^2 + \left\| w_n(t) \right\|_{Q^t_n}^2 + \left( \alpha + \frac{1}{2} \beta L \right) \left\| w_n(t) \right\|_{Q^t_n}^2. \]  

(48)

Let us estimate some terms from the right-hand part of relation (48). For this we will have to make repeated use of (13).

By (32), (33) and (36) we get

\[ \left\| w_n(t) \right\|_{Q^t_n}^2 \leq \left\| w_n'(t) \right\|_{Q^t_n}^2 + \Phi_n(t) \leq c_3. \]  

(50)

From (16), (17), (23) and (31) follows

\[ \left\| p_n w(t) \right\|_{Q^t_n}^2 - \left\| w_n(t) \right\|_{Q^t_n}^2 \]
\[ \leq \left( \frac{\pi}{L} \right)^2 \sum_{j=1}^n j^2 \left| w_n(t) - w_m(t) \right| \]
\[ \leq \sqrt{2} \left( \left\| p_n w(t) \right\|_{Q^t_n}^2 + \left\| w(t) \right\|_{Q^t_n}^2 \right) \| \Delta w_n(t) \|_{Q^t_n} \]
\[ \leq \sqrt{2}(c_1 + c_2) \| \Delta w_n(t) \|_{Q^t_n}. \]  

(51)

Finally, again using (23) we find

\[ \left\| Q_n^2 p_n w(t), (\Delta w_n(t)) \right\|_{Q^t_n} \]
\[ \leq \left\| p_n w(t) \right\|_{Q^t_n} \left\| \Delta w_n(t) \right\|_{Q^t_n} \leq c_0 \left\| \Delta w_n(t) \right\|_{Q^t_n}. \]  

(52)

Relations (48)-(52) together with (13), (22) and (39) allow us to conclude that

\[ F_n(t) = \int_0^t F_n'(\tau) d\tau \leq \max(c_5, c_6) \int_0^t F_n(\tau) d\tau, \]
where

\[ c_5 = 1 + v, \quad c_6 = \left( \alpha + \left( \frac{\pi}{L} \right)^2 \right) \left( v + \frac{1}{2} c_3 \beta L \right), \]  

(53)

By (32), (33) and (36) we get

\[ \left\| w_n(t) \right\|_{Q^t_n}^2 \leq \left\| w_n'(t) \right\|_{Q^t_n}^2 + \Phi_n(t) \leq c_3. \]  

(49)

Applying the Gronwall inequality and definition (49), we obtain the proven inequality (47) together with the formula for the coefficient \( c(t) \)

\[ c(t) = c_4 \sqrt{T e^{|\Phi_n(0)|}}, \quad \Box \]

Note that if we weaken the accuracy requirement, relations (53) can be simplified. By virtue of (38) we can take \( c_1 \) instead of \( c_2 \) and replace the value \( \Phi_n(0) \) contained in \( c_3 \) by one of its upper bounds from (37).

VI. SOLUTION OF THE GALERKIN SYSTEM

Here we consider a method of solving the system (11), (12). Let us introduce, on the time segment \([0, T]\), a grid with step \( \tau = T/M \) and nodes \( t_m = m \tau, \ m = 0, 1, \ldots, M \). An approximate value of \( w_m(t_m) \) denoted by \( w_m^m \) is determined by a difference scheme of the form

\[ w_{ni, m}^{m-1} + \left( \frac{\pi a}{L} \right)^2 \frac{w_{ni, m}^m + w_{ni, m}^{m-2}}{2} + \left( \frac{\pi a}{L} \right)^2 \]
\[ \times \left( \alpha + \frac{\pi^2}{2L} \sum_{j=1}^n j^2 \left( w_{nj, m}^m \right)^2 + \left( w_{nj, m}^{m-2} \right)^2 \right) w_{nj, m}^m + w_{nj, m}^{m-2} \]
\[ = \frac{1}{2} (f_{i, m}^m + f_{i, m}^{m-2}), \]
\[ i = 1, 2, \ldots, n, \quad m = 2, 3, \ldots, M, \]
with the conditions

\[ w_{ni, 0}^0 = a_i^{(0)}, \]
\[ w_{ni, 1}^1 = a_i^{(0)} + \tau a_i^{(1)} = \left( \frac{\pi a}{L} \right)^2 + \left( \frac{\pi a}{L} \right)^2 \]
\[ \times \left( \alpha + \frac{\pi^2}{2L} \sum_{j=1}^n j^2 a_j^{(0)} \right) a_i^{(0)}, \]

(55)

where \( f_{i, m-1}^l = f_{i, m-1}, \ l, m = 0, 2, \ldots \).

From (54) and (55) it follows that if the counting is performed from level to level, then, knowing the results for
the preceding levels, at the $m$th time level, $m = 2, 3, \ldots, M$, i.e. for $t = t_m$, we have to solve a system of nonlinear equations with respect to $w_{ni}^m$, $i = 1, 2, \ldots, n$, which has the form

$$\begin{align*}
&\left[1 + \frac{r_i^2}{2} \left(\frac{\pi}{L}\right)^2 \alpha + \left(\frac{\pi}{L}\right)^2 \right]
+ \beta \frac{\pi^2}{4L} \sum_{j=1}^{n} j \left[\left(w_{nj}^m \right)^2 + \left(w_{nj}^{m-2} \right)^2 \right] \left(w_{ni}^m + w_{ni}^{m-2} \right) \\
&= 2w_{ni}^{m-1} + \frac{\tau^2}{2} \left(f_i^m + f_i^{m-2} \right), \\
i &= 1, 2, \ldots, n.
\end{align*}$$

System (56) is solved by the iteration method consisting in calculating successive approximations by Jacobi’s rule [19]

$$\begin{align*}
&\left[1 + \frac{r_i^2}{2} \left(\frac{\pi}{L}\right)^2 \alpha + \left(\frac{\pi}{L}\right)^2 \right]
+ \beta \frac{\pi^2}{4L} \sum_{j=1}^{n} j \left[\left(w_{nj,k}^m \right)^2 + \left(w_{nj,k}^{m-2} \right)^2 \right] \left(w_{ni,k}^m + w_{ni,k}^{m-2} \right) \\
&= 2w_{ni,k}^{m-1} + \frac{\tau^2}{2} \left(f_i^m + f_i^{m-2} \right), \\
i &= 1, 2, \ldots, n, \\
k &= 0, 1, \ldots.
\end{align*}$$

where $w_{ni,k}^m$ and $w_{ni,k}^{m-1}$ are the $k$th and the final iteration approximation of $w_{ni}^m$ and $w_{ni}^{m-1}$, $i = 1, 2$.

For fixed $i$, (57) is a cubic equation with respect to $w_{ni,k+1}$. The Cardano formula [15] allows us to determine $w_{ni,k+1}^m$ in an explicit form. We get

$$i w_{ni,k+1}^m = -\frac{i w_{ni,k}^{m-2}}{3} + \frac{1 - \sqrt{-3}}{2} \sum_{j=1}^{L} \left[ \frac{S_j}{2} \left( \frac{S_j^2}{4} + \frac{r_i^4}{27} \right)^{\frac{1}{3}} \right],$$

$$k = 0, 1, \ldots, i = 1, 2, \ldots, n,$

where

$$r_i = q_i + \frac{2}{3} \left( i w_{ni,k}^{m-2} \right)^2 + \frac{1}{\tau^2 i^2 \pi \beta} \left( \frac{2L}{\tau} \right)^3,$$

$$s_i = \frac{2}{3} i w_{ni,k}^{m-2} \left(q_i + \frac{10}{9} \left(i w_{ni,k}^{m-2} \right)^2 \right) - \frac{1}{\tau^2 i^2 \pi \beta} \left( \frac{2L}{\tau} \right)^3 \left[ -\frac{2 i w_{ni,k}^{m-2}}{3} + 2 i w_{ni,k}^{m-1} + \frac{\tau^2}{2} \right] \left( f_i^m + f_i^{m-2} \right),$$

$$q_i = \frac{4L}{\pi^2 \beta} \left( \alpha + \left(\frac{\pi}{L}\right)^2 \right) + \sum_{j=1, j \neq i}^{n} j \left[\left(w_{nj,k}^m \right)^2 + \left(w_{nj,k}^{m-2} \right)^2 \right].$$

Thus the proposed algorithm is reduced to the calculation by formula (58). Having $w_{ni,k}^m$, we can construct the series

$$\sum_{i=1}^{n} w_{ni,k}^m \sin \frac{i \pi x}{L},$$

which gives an approximate value of the exact solution $w(x,t)$ of problem (1), (2) for $t = t_m$.

The case where $f(x,t) = 0$ in (1) was considered in the author’s paper [21].

REFERENCES


